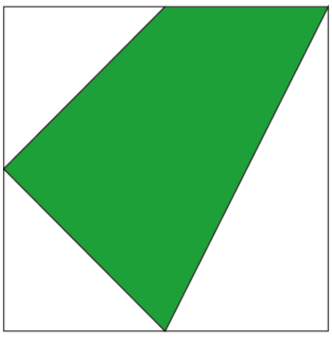
Maths Challenge - Week 301 – Solutions

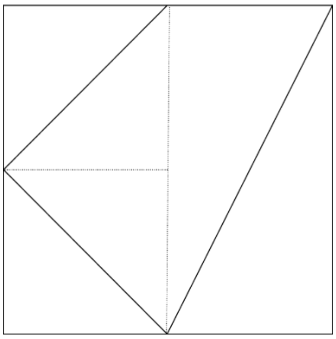
Welcome to week 301 of our weekly maths challenge, with problems and puzzles posed by David Browning, Rod Marshall, Ian Stewart, Annie Stothers and the u3a Maths and Stats Subject Adviser - David Martin. If you would like to share your ideas on how to solve these puzzles please join our learning forum or discuss within your u3a and interest group. Check back each week for the solutions and let us know how you get on by contacting the u3a office. New maths puzzles will go up onto the website every Thursday.

Question 1.



What fraction of the square is occupied by the central quadrilateral which has three of its vertices at midpoints of the sides of the square?

Solution



The dotted lines illustrate that the three outer triangles are 1/8, 1/8 and 1/4 the area of the square, so the area of the central quadrilateral is 1/2 of the area of the square.

Question 2.

Fiona measured the interior angles of a regular polygon and found that they were 150°. What polygon was she measuring?

Solution

An n sided regular polygon has interior angles of $\frac{n-2}{n}180°$

So, $\frac{n-2}{n}180 = 150$ i.e., $180n - 360 = 150n$

$30n = 360$

$n = 12$

The regular polygon was a dodecagon.

Question 3.

In the game of bridge, each of the four players is dealt 13 cards. Each player usually counts the number of honour points in their hand with four points for an ace, three for a king, two for a queen and one for a jack. If a player has two aces, two kings and one jack in the first 11 cards that they look at, what is the probability that their 13 cards will have exactly 20 honour points?

Solution

The first 11 cards have 15 honour points so 20 honour points will be achieved if the final two cards are either an ace and a jack or a king and a queen.

The final two cards can be any of 41 cards and those 41 cards contain two aces, two kings, four queens and three jacks.

The probability that the 12th card is an ace is 2/41 and the probability that the 13th card is then a jack is 3/40. So, the probability of the 12th card being an ace and the 13th card being a jack is (2/41) x (3/40) = 2 x 3/(41 x 40). The probability that the 12th card is a jack is 3/41 and the probability that the 13th card is then an ace is 2/40. So, the probability of the 12th card being an ace and the 13th card being a jack is (3/41) x (2/40) = 2 x 3/(41 x 40) So, the probability that the player has an ace and jack in their two final cards is 2 x 2 x 3/(41 x 40) = 12/(41 x 40).

Applying the same logic to the final two cards being a king and queen shows that the probability of this outcome is (2 x 2 x 4)/(41 x 40) = 16/(41 x 40).

The probability that the honour point count for the final two cards is 5 giving an honour point count for the full hand of 20 points is (12 + 16)/(41 x 40) = 0.017 to three decimal places.

Question 4.

A simple model for the deceleration of a high-speed train can be expressed in the form $v_n = av_{n-1}, n = 1, 2, 3$ ... where v_n is the speed after n time steps, a is a constant and the calculations are performed at fixed time steps Δt. Using this formula with $a = 0.75$ and $\Delta t = 5$ seconds, how long would it take for a train to decelerate from a cruising speed of 300 km/h to a crawling speed of 10 ± 1 km/h and how far would the train travel in this time?

Solution

The speed of the train after each time increment can be expressed in the form:

$$v_1 = av_0$$
$$v_2 = av_1 = a(av_0) = a^2v_0$$
$$v_3 = av_2 = a(a^2v_0) = a^3v_0$$
$$\dots$$

This can be generalised into the formula $v_n = a^n v_0$

Rearranging, taking natural logarithms and substituting the values for $v_0 = 300$, $v_n = 10$ and $a = 0.75$:

$$a^n = \frac{v_n}{v_0}$$
$$ln\ (a^n) = ln\ \left(\frac{v_n}{v_0}\right)$$
$$n\ ln(a) = ln\left(\frac{v_n}{v_0}\right)$$
$$n = \frac{ln\left(\frac{v_n}{v_0}\right)}{ln(a)} = \frac{-3.40120}{-0.28768} = 11.823$$

Given that n is required to be an integer, v_{11} and v_{12} can be calculated as:

$v_{11} = 300\ a^{11} = 12.671$ km/h
$v_{12} = 300\ a^{12} = 9.503$ km/h

Therefore, a total of 60 seconds (12 steps of 5 seconds) is required for the train speed to reduce from 300 to 10 ± 1 km/h.

The total distance d_t travelled whilst slowing down can be approximated for small Δt by summing the product of the average speed for each time step and the length of the time step Δt:

$$d_t = \Delta t\left(\frac{v_0 + v_1}{2} + \frac{v_1 + v_2}{2} + \cdots \frac{v_{n-1} + v_n}{2}\right)$$
$$d_t = \Delta t\left(\frac{v_0}{2} + v_1 + v_2 + \cdots v_{n-1} + \frac{v_n}{2}\right) \text{(for small}$$
$$d_t = \Delta t\ v_0\left(\frac{1}{2} + a + a^2 + \cdots + a^{n-1} + \frac{a^n}{2}\right)$$

The expression $a + a^2 + \cdots a^{n-1}$ can be identified as a finite geometric expression and can be represented in the concise form:

$$a + a^2 + \cdots + a^{n-1} = \frac{a(1 - a^{n-1})}{1 - a}, \qquad a \neq 1$$

Therefore, the total distance travelled is given by:

$$d_t = \Delta t\ v_0\left(\frac{1}{2} + \frac{a(1 - a^{n-1})}{1 - a} + \frac{a^n}{2}\right)$$

Substituting for $\Delta t = \frac{5}{3600}$ h, $v_0 = 300$ km/h and $a = 0.75$ and evaluating for $n = 12$:

$$d_t = \frac{5}{3600} \times 300\left(\frac{1}{2} + \frac{0.75 \text{ x } 0.957765}{0.25} + 0.015838\right) = 1.412 \text{ to 3 decimal places}$$

Therefore, the total distance travelled in 60 seconds is 1.4 km (to 1 decimal pace).