Week Fifty Problems and Solutions

Question 1.

Can you find a way of placing the digits 3, 5, 7 and 9 in a square grid in such a way that all four of the two-digit numbers reading across and down are prime?

Solution

No number ending in 5 (other than 5 itself) can be prime, so the 5 must go in the NW (north-west) box: $\begin{matrix} 5 & ? \\ ? & ? \end{matrix}$

57 is not a prime, so, 7 must go in the SE corner: $\begin{matrix} 5 & ? \\ ? & 7 \end{matrix}$

37 and 97 are both prime, so we have the two solutions: $\begin{matrix} 5 & 3 \\ 9 & 7 \end{matrix}$ and $\begin{matrix} 5 & 9 \\ 3 & 7 \end{matrix}$

Question 2.

Which is the better fit – a round peg in a square hole or a square peg in a round hole?

In this question, “better fit” means filling more of the available space.

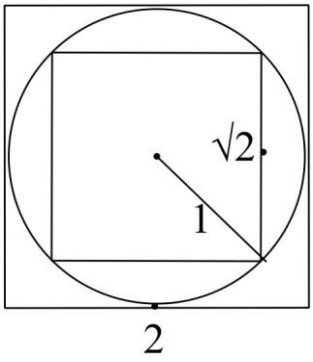
Solution

Consider a square hole of side 2. Its area is 4 square units.

A circle of radius 1 will fit snugly in the square. The area of the circle is simply π square units, so a round peg will fill π/4 ≈ 78.54% of a square hole.

Now consider a square inside the circle.

As the square has a diagonal of length 2, its sides must be of length √2 (by Pythagoras) and so its area is 2 square units. Hence a square peg fills 2/π ≈ 63.66% of a round hole.



A round peg in a square hole is therefore the better fit.

Question 3.

John has 12 more cattle than Bill. George has twice the total number of cattle which John and Bill have. ½ of Bill's cattle, ⅔ of John's cattle and ¾ of George's cattle give birth to a calf. Each birth is of a single calf. There are now 101 calves. How many cattle did Bill have before calving began?

Solution

If C is the number of cattle that Bill has before calving began, John has C+12 cattle and George has 4C+24 cattle.

The number of calves born is therefore ½C + ⅔(C+12) + ¾(4C+24) = 25C/6 + 26.

As there are 101 calves, 25C/6 +26 = 101 so 25C/6 = 75 giving C = 18. Bill had 18 cattle before calving began.

Question 4.

Every 10 seconds, a cell sub-divides and becomes two cells. Immediately after every second sub-division, ¼ of all cells die. How long will it take for one cell to increase to over 1 million cells?

Solution

After 10 seconds, 1 cell becomes 2 cells. After 20 seconds, those 2 cells become 4 cells but as ¼ of the cells then die, there are 3 cells. The same process applies over the next 20 seconds with each cell at the beginning of the 20 second period turning into 3 cells at the end of the 20 second period. It follows that after n periods of 20 seconds, the number of cells will be 3^n.

3^{12} = 531,441 so after 12 periods of 20 seconds, there will be 531,441 cells. 10 seconds later, this number will double to 1,062,882. So, there will be over 1 million cells after 12 periods of 20 seconds plus 10 seconds, i.e., after 4 minutes 10 seconds.